Large Sample Techniques for Statistics
Chapter 8: Martingales

Di SU

Department of Statistics, The Chinese University of Hong Kong

Summer Lab Meeting

June 15, 2021

Overview

8.1 Introduction

Definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let I represent an index set of integers. Let $\mathcal{F}_n, n \in I$ be a nondecreasing sequence of σ-fields of $\mathcal{F}$ sets.

(Discrete-Time) Martingale, Submartingale, Supermartingale

A sequence of random variables $S_n, n \in I$, is called a (discrete-time) martingale with respect to $\mathcal{F}_n, n \in I$, or $S_n, \mathcal{F}_n, n \in I$, is a martingale if it satisifies:

$$S_n \in \mathcal{F}_n, E(S_m|\mathcal{F}_n) = S_n \text{ a.s. } \forall m, n \in I, m > n; \tag{1}$$

or equivalently

$$S_n \in \mathcal{F}_n, E(S_{n+1}|\mathcal{F}_n) = S_n \text{ a.s. } \forall n \text{ such that } n, n+1 \in I. \tag{2}$$

Similarly, a sequence $S_n, \mathcal{F}_n, n \in I$ is a submartingale (supermartingale) if the equality in (1) is changed to $\geq$ ($\leq$).

Definitions

Martingale Differences

Let $X_n, n \in I$, be a sequence of random variables. We say $X_n, \mathcal{F}_n, n \in I$, is a sequence of martingale differences if

$$X_n \in \mathcal{F}_n, E(X_{n+1}|\mathcal{F}_n) = 0 \text{ a.s.} \tag{3}$$

for all n such that $n, n+1 \in I$.

Definitions

A sequence of random variables $\xi_n, n \in I$, is said to be adapted to $\mathcal{F}_n, n \in I$, if $\xi_n \in \mathcal{F}_n, n \in I$. A sequence $\eta_n, n \in I$, is said to be predictable with respect to $\mathcal{F}_n, n \in I$, if $\eta_n \in \mathcal{F}_{n-1}, n-1, n \in I$

Definitions

Stopping Time

A measurable function τ taking values in $\{1, 2, \ldots, \infty\}$ is called a stopping time with respect to $\mathcal{F}_n, n \geq 1$, if $\{\tau = n\} \in \mathcal{F}_n, n \geq 1$. It means we can know whether to stop at time n based on information we have so far.
For each stopping time τ we can define a corresponding σ -field

$$\mathcal{F}_\tau = \{A \,| \in \mathcal{F}_\infty : A \cap \{\tau = n\} \in \mathcal{F}_n, n \geq 1\}$$

where $\mathcal{F}_\infty = \sigma\left(\cup_{n=1}^{\infty} \mathcal{F}_n\right)$.

Properties

Lemma

(Relation between martingales and martingale differences)

- *If* $X_n, \mathcal{F}_n, n \geq 1$, *is a sequence of martingale differences, then* $S_n = \sum_{i=1}^n X_i, \mathcal{F}_n, n \geq 1$, *is a martingale. Conversely, if* $S_n, \mathcal{F}_n, n \geq 1$, *is a martingale, define* $X_1 = S_1$ *and* $X_n = S_n - S_{n-1}, n \geq 2$, *then* $X_n, \mathcal{F}_n$, $n \geq 1$, *is a sequence of martingale differences.*
- *(i) If* $\xi_n, n \geq 1$ *is adapted to* $\mathcal{F}_n, n \geq 1$, *let* $X_1 = \xi_1$, $X_n = \xi_n - \mathrm{E}\left(\xi_n \mid \mathcal{F}_{n-1}\right), n \geq 2$. *Then* $X_n, \mathcal{F}_n, n \geq 1$, *is a sequence of martingale differences; hence,* $S_n = \sum_{i=1}^n X_i, \mathcal{F}_n, n \geq 1$, *is a martingale.*
  (ii) If $X_n, \mathcal{F}_n, n \in I$, *is a sequence of martingale differences, and* $\eta_n, n \in I$, *is predictable with respect to* $\mathcal{F}_n, n \in I$, *then* $\eta_n X_n, \mathcal{F}_n, n \in I$, *is a sequence of martingale differences.*

Properties

Lemma

(Operations preserving the property)

- *(i) If* $X_n^{(j)}, \mathcal{F}_n, n \in I, j = 1, 2$*, are two sequences of martingale differences, then* $X_n^{(1)} + X_n^{(2)}, \mathcal{F}_n, n \in I$*, is a sequence of martingale differences.*
  (ii) If $S_n^{(j)}, \mathcal{F}_n, n \in I, j = 1, 2$*, are two martingales (submartingales, supermartingales), then* $S_n^{(1)} + S_n^{(2)}, \mathcal{F}_n, n \in I$*, is a martingale (submartingale, supermartingale).*
  (iii) If $S_n^{(j)}, \mathcal{F}_n, n \in I, j = 1, 2$*, are two submartingales, then* $S_n^{(1)} \vee S_n^{(2)}$*,* $\mathcal{F}_n, n \in I$*, is a submartingale.*
  (iv) If $S_n^{(j)}, \mathcal{F}_n, n \in I, j = 1, 2$*, are two supermartingales, then* $S_n^{(1)} \wedge S_n^{(2)}$*,* $\mathcal{F}_n, n \in I$*, is a supermartingale.*

Properties

Lemma

(Useful transformations)

- *(i) If* $S_n, \mathcal{F}_n, n \in I$ *is a martingale, then for any convex (concave) function* $\psi, \psi(S_n), \mathcal{F}_n, n \in I$, *is a submartingale (supermartingale). e.g.* S_n^2.
  (ii) If $S_n, \mathcal{F}_n, n \in I$, *is a submartingale (supermartingale), then for any nondecreasing convex (concave) function* $\psi, \psi(S_n), \mathcal{F}_n, n \in I$, *is a submartingale (supermartingale). e.g.* e^{S_n}.
  (iii) If $S_n, \mathcal{F}_n, n \in I$, *is a supermartingale (submartingale), then for any nonincreasing convex (concave) function* $\psi, \psi(S_n), \mathcal{F}_n, n \in I$, *is a submartingale (supermartingale). e.g.* $-X_i$.

Properties

Lemma

- *Suppose that* $X_i, \mathcal{F}_i, 1 \le i \le n$, *is a sequence of martingale differences. Then the following hold.*
  (i) $\mathrm{E}(X_i) = 0, 2 \le i \le n$
  (ii) If $\mathrm{E}\left(X_i^2\right) < \infty, 1 \le i \le n$, *then the sequence* $X_i, 1 \le i \le n$, *is orthogonal; that is,* $\mathrm{E}(X_i X_j) = 0$, *if* $i \ne j$. *It follows that*

$$\mathrm{E}\left(\sum_{i=1}^{n} X_i\right)^2 = \sum_{i=1}^{n} \mathrm{E}\left(X_i^2\right)$$

  Example: P244

Properties

Lemma

- *Suppose that τ is a stopping time with respect to $\mathcal{F}_n, n \geq 1$.*
  (i) $\mathcal{F}_\tau$ is a σ-field and $\tau \in \mathcal{F}_\tau$.
  (ii) If $S_n, n \geq 1$, is adapted to $\mathcal{F}_n, n \geq 1$, and S_∞ is defined as $\limsup S_n$, then $S_\tau \in \mathcal{F}_\tau$ Now suppose that τ_1 and τ_2 are stopping times with respect to $\mathcal{F}_n, n \geq 1$.
  (iii) $\tau_1 \vee \tau_2$ and $\tau_1 \wedge \tau_2$ are both stopping times with respect to $\mathcal{F}_n, n \geq 1$.
  (iv) If $\tau_1 \leq \tau_2$, then $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$

8.2 The Optional Stopping Theorem

Theorem 8.1.

Theorem

Let $S_n, \mathcal{F}_n, n \geq 1$, *be a submartingale, and let* τ_2 *be a stopping time with respect to* $\mathcal{F}_n, n \geq 1$, *such that* $\mathrm{P}(\tau_2 < \infty) = 1$ *and* $\mathrm{E}(S_{\tau_2})$ *exists. If*

$$\liminf \mathrm{E}\left\{S_n^+ 1_{(\tau_2 > n)}\right\} = 0,$$

then for any stopping time τ_1 *with respect to* $\mathcal{F}_n, n \geq 1$, *as long as* $\mathrm{E}\left\{S_{\tau_1} 1_{(\tau_1 \leq \tau_2)}\right\}$ *exists, we have*

$$\mathrm{E}(S_{\tau_2} \mid \mathcal{F}_{\tau_1}) \geq S_{\tau_1} \text{ a.s. } \{\tau_1 \leq \tau_2\}.$$

Generalization

Let $x^+ = x$ if $x \geq 0$ and $x^+ = 0$ otherwise. Denote $x^- = (-x)^+$. We have $|x| = x^+ + x^-$.

Theorem

Let $S_n, \mathcal{F}_n, n \geq 1$, be a martingale (resp. supermartingale), and let τ_2 be a stopping time with respect to $\mathcal{F}_n, n \geq 1$, such that $\mathrm{P}(\tau_2 < \infty) = 1$ and $\mathrm{E}(S_{\tau_2})$ exists. If

$$\liminf \mathrm{E}\left\{|S_n| 1_{(\tau_2 > n)}\right\} = 0 \ \ (\textit{resp.} \liminf \mathrm{E}\left\{S_n^- 1_{(\tau_2 > n)}\right\} = 0),$$

then for any stopping time τ_1 with respect to $\mathcal{F}_n, n \geq 1$, as long as $\mathrm{E}\left\{S_{\tau_1} 1_{(\tau_1 \leq \tau_2)}\right\}$ exists, we have

$$\mathrm{E}\left(S_{\tau_2} \mid \mathcal{F}_{\tau_1}\right) = (\textit{resp.} \leq)\ S_{\tau_1} \ \ \textit{a.s.} \ \ \{\tau_1 \leq \tau_2\}.$$

For a non-decreasing sequence of stopping times, $\tau_1 \leq \tau_2 \leq \ldots$, $S_{\tau_k}, \mathcal{F}_{\tau_k}, k \geq 1$ remains a martingale (or respectively submartingale or supermartingale).

8.3 The Martingale Convergence Theorem

Theorem 8.2.

Theorem

Suppose that $S_n, \mathcal{F}_n, n \geq 1$, *is a submartingale such that*

$$\sup_{n\geq 1} \mathrm{E}\left(S_n^+\right) < \infty$$

Then $S_\infty = \lim_{n\to\infty} S_n$ *exists almost surely, and it has the following properties: (i)* $\mathrm{E}\left(S_\infty^+\right) < \infty$; *(ii)* $|S_\infty| < \infty$ *a.s.* $\{S_1 > -\infty\}$; *and (iii) if*

$$\mathrm{E}\left(|S_n|\right) < \infty, n \geq 1 \;\; (i.e. \sup_{n\geq 1} E(|S_n|) < \infty),$$

then $\mathrm{E}\left(|S_\infty|\right) < \infty$.

- The L^1 boundedness of a martingale is all that is needed for the a.s. convergence of the submartingale in the usual sense.
- Analogy: Bounded monotone sequences of real numbers.

Extension & Application

Corollary 8.2.

Let $S_n, \mathcal{F}_n, n \geq 1$ be a nonnegative supermartingale. Then S_n converges almost surely to a limit S_∞. Furthermore, if $\mathrm{E}(S_1) < \infty$, then $\mathrm{E}(|S_\infty|) < \infty$.

For any $a < b$, let $U_n(a, b)$ denote the number of times that $S_1, S_2, \ldots, S_n$ crosses from a value $\leq a$ to one $\geq b$ (known as upcrossing).

Lemma 8.8. (Doob 1960)

Suppose that $S_k, \mathcal{F}_k, 1 \leq k \leq n$, is a submartingale. Then for any $a < b$, we have

$$\mathrm{E}\{U_n(a, b)\} \leq \frac{\mathrm{E}\{(S_n - a)^+\}}{b - a}$$

8.4 Martingale Laws of Large Numbers

A weak law of large numbers

Theorem 8.3. (Sufficient conditions)

Let $S_n = \sum_{i=1}^{n} X_i, \mathcal{F}_n, n \geq 1$, be a martingale. Let the sequence of normalizing constants $0 < a_n \uparrow \infty$. Then

$$a_n^{-1} S_n \xrightarrow{\mathrm{P}} 0 \text{ as } n \to \infty$$

if the following conditions hold:

(i) $\sum_{i=1}^{n} \mathrm{P}\left(|X_i| > a_n\right) \longrightarrow 0$

(ii) $a_n^{-1} \sum_{i=1}^{n} \mathrm{E}\left\{X_i 1_{(|X_i| \leq a_n)} \mid \mathcal{F}_{i-1}\right\} \xrightarrow{\mathrm{P}} 0$

(iii) $a_n^{-2} \sum_{i=1}^{\bar{n}} \mathrm{E}\left[\operatorname{var}\left\{X_i 1_{(|X_i| \leq a_n)} \mid \mathcal{F}_{i-1}\right\}\right] \longrightarrow 0$

- The three conditions are sufficient, but not necessary in the martingale case (Example 8.12).
- A possible choice of a_n: $a_n = n$.

Some strong laws of large numbers

For any nondecreasing sequence of positive numbers a_n such that $a_n \to \infty$, by Kronecker's lemma, if the series

$$\sum_{i=1}^{\infty} \frac{X_i}{a_i} \tag{4}$$

converges, then

$$\frac{S_n}{a_n} \to 0. \tag{5}$$

The problem is when (4) converges.

Some strong laws of large numbers

Theorem 8.4.

For any $1 \le p \le 2$, the series (4) converges and (5) holds a.s. on the set $\left\{\sum_{i=1}^{\infty} a_i^{-p} \mathrm{E}\left(|X_i|^p \mid \mathcal{F}_{i-1}\right) < \infty\right\}$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Theorem 8.5.

(i) Let $X_i, i \ge 1$, be any sequence of random variables. Then, the conclusion of Theorem 8.4 holds for any $p \in (0, 1)$.
(ii) Let $X_i, \mathcal{F}_i, i \ge 1$, be a sequence of martingale differences. For any $p > 2$ and any sequence $b_i > 0, i \ge 1$, such that $\sum_{i=1}^{\infty} b_i < \infty$, (4) converges and (5) holds a.s. on $\left\{\sum_{i=1}^{\infty} a_i^{-p} b_i^{1-p/2} \mathrm{E}\left(|X_i|^p \mid \mathcal{F}_{i-1}\right) < \infty\right\}$

- The two theorems still hold if a_n is a sequence of predictable random variables w.r.t. $\mathcal{F}_n, n \ge 1$, provided that $a_n > 0$ and $a_n \to \infty$.

Some strong laws of large numbers

SLLN for martingales

$n^{-1} S_n \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$ provided either

$$\sum_{i=1}^{\infty} \frac{\mathrm{E}\left(|X_i|^p\right)}{i^p} < \infty$$

for some $1 \leq p \leq 2$ or

$$\sum_{i=1}^{\infty} \frac{\mathrm{E}\left(|X_i|^p\right)}{i^p b_i^{p/2-1}} < \infty$$

for some $p > 2$ and $b_i > 0, i \geq 1$, such that $\sum_{i=1}^{\infty} b_i < \infty$.

Some strong laws of large numbers

Theorem 8.6. Another SLLN for martingales

Let $S_n = \sum_{i=1}^n X_i, \mathcal{F}_n, n \geq 1$ be a martingale. If for some $p \geq 1$, we have

$$\sum_{i=1}^{\infty} \frac{\mathrm{E}\left(|X_i|^{2p}\right)}{i^{p+1}} < \infty$$

then $n^{-1} S_n \overset{\text{a.s.}}{\to} 0$ as $n \to \infty$.

The conditions in the two SLLNs don't imply each other.

8.5 A Martingale Central Limit Theorem

Definitions

Array of martingales and martingale differences

An array of martingales is an array $S_{ni} = \sum_{j=1}^{i} X_{nj}, \mathcal{F}_{ni}, 1 \leq i \leq k_n, n \geq 1$, such that for each $n, S_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq k_n$, is a martingale, where k_n is a nondecreasing sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$. A special choice is $k_n = n$.

Assume that S_{ni} has mean 0 and a finite second moment for all n and i. Define $S_{n0} = 0$ and $\mathcal{F}_{n0} = \{\emptyset, \Omega\}$. Then for each $n, X_{ni}, \mathcal{F}_{ni}, 1 \leq i \leq k_n$, is a sequence of martingale differences with $E(X_{ni}) = 0$ and $E(X_{ni}^2) < \infty, 1 \leq i \leq k_n$.

Stable convergence

Let $Y_n, n \geq 1$, be a sequence of random variables on the probability space $(\Omega, \mathcal{F}, P)$ converging in distribution to a random variable Y. We say the convergence is stable, denoted by $Y_n \xrightarrow{P} Y$ (stably), if for all continuity points y of Y and all $A \in \mathcal{F}$, the limit $\lim_{n\to\infty} \mathrm{P}(\{Y_n \leq y\} \cap A) = P_y(A)$ exists and $P_y(A) \to \mathrm{P}(A)$ as $y \to \infty$.

CLT

Let $X_{ni}, \mathcal{F}_{ni}, 1 \le i \le k_n, n \ge 1$, be an array of martingale differences as above. Suppose that

$$\max_{1\le i\le k_n} |X_{ni}| \xrightarrow{\mathrm{P}} 0, \quad \sum_{i=1}^{k_n} X_{ni}^2 \xrightarrow{\mathrm{P}} \eta^2$$

where η^2 is a random variable, and $\mathrm{E}\left(\max_{1\le i\le k_n} X_{ni}^2\right)$ is bounded in n. In addition, assume that the σ-fields satisfy

$$\mathcal{F}_{ni} \subset \mathcal{F}_{(n+1)i}, \quad 1 \le i \le k_n, n \ge 1$$

Then we have, as $n \to \infty$,

$$S_{nk_n} = \sum_{i=1}^{k_n} X_{ni} \xrightarrow{\mathrm{d}} Z \text{ (stably)}$$

where the random variable Z has characteristic function

$$c_Z(t) = \mathrm{E}\left\{\exp\left(-\eta^2 t^2/2\right)\right\}.$$

CLT

- When $\eta = 1$, $Z \sim N(0, 1)$.
- If η is a constant, we don't require $\mathcal{F}_{ni} \subset \mathcal{F}_{(n+1)i}, \quad 1 \leq i \leq k_n, n \geq 1$.
- The convergence form of martingale array is more convenient as in practice, the observations under a smaller sample size may not be a subset of those under a larger one. Moreover, normalization of the sequence can be made more expicit.

Convergence Rate

Uniform convergence rate

Let $S_{ni} = \sum_{j=1}^{i} X_{nj}, \mathcal{F}_{ni}, 1 \le i \le n$, be an array of martingales, where $\mathcal{F}_{ni} = \sigma\left(X_{n1}, \ldots, X_{ni}\right), 1 \le i \le n$. Let $V_{ni}^2 = \sum_{j=1}^{i} \mathrm{E}\left(X_{nj}^2 \mid \mathcal{F}_{nj-1}\right), 1 \le i \le n$. Write $S_n = S_{nn}$ and $V_n^2 = V_{nn}^2$. If

$$\max_{1 \le i \le n} |X_{ni}| \le \frac{M}{\sqrt{n}},$$

$$\mathrm{P}\left\{\left|V_n^2 - 1\right| > 9M^2 D \frac{(\log n)^2}{\sqrt{n}}\right\} \le B \frac{\log n}{n^{1/4}}$$

for some constants M, B, and D with $D \ge e$, then for $n \ge 2$, we have

$$\sup_{-\infty < x < \infty} \left|\mathrm{P}\left(S_n \le x\right) - \Phi(x)\right| \le c \frac{\log n}{n^{1/4}}$$

where Φ is the cdf of $N(0,1)$ and $c = 2 + B + 7M\sqrt{D}$

Convergence Rate

- It is shown that $n^{-1/4}\log n$ is the best possible rate for martingales
- The slower convergence rate is due to the dependence of martingale differences. The dependence reduces the effective sample size.
- A special choice of S_{ni} : $S_{ni} = \sum_{j=1}^{i} X_i/\sqrt{n}$ where X_i may be a time series.

Convergence Rate

Nonuniform convergence rate

With the same notation as above and $U_n^2 = \sum_{i=1}^n X_{ni}^2$, define, for any $0 < \delta \le 1$

$$p_n = \sum_{i=1}^n \mathrm{E}\left\{|X_{ni}|^{2(1+\delta)}\right\} + \mathrm{E}\left(\left|U_n^2 - 1\right|^{1+\delta}\right)$$

$$q_n = \sum_{i=1}^n \mathrm{E}\left\{|X_{ni}|^{2(1+\delta)}\right\} + \mathrm{E}\left(\left|V_n^2 - 1\right|^{1+\delta}\right)$$

There is a constant c_δ depending only on δ such that for all x,

$$|\mathrm{P}(S_n \le x) - \Phi(x)| \le c_\delta \frac{(p_n \wedge q_n)^{1/(3+2\delta)}}{1 + |x|^{4(1+\delta)^2/(3+2\delta)}}$$

Note that it is possible that $c_\delta \to \infty$ as $\delta \to 0$.

8.6 Convergence Rate in SLLN and LIL

Convergence rate of SLLN

Let $\sigma_n^2 = \sum_{i=1}^n E(X_i^2)$. Suppose that there are constants $b_i > 0$ such that $\mathrm{E}\left(X_i^2 \mid \mathcal{F}_{i-1}\right) \le b_i$ a.s. and constants $0 < c_1 < c_2$ such that

$$c_1\sigma_n^2 \le \sum_{i=1}^n b_i \le c_2\sigma_n^2$$

Furthermore, suppose that

$$\sup_{-\infty<x<\infty} \left|\mathrm{P}\left(S_n < x\sigma_n\right) - \Phi(x)\right| \longrightarrow 0, \text{ as } n \to \infty.$$

- Uniform convergence in CLT is required
- $\sup_{-\infty<x<\infty} \left|\mathrm{P}\left(S_n < x\sigma_n\right) - \Phi(x)\right| \longrightarrow 0 \iff$ $\sup_{-\infty<x<\infty} \left|\mathrm{P}\left(S_n \le x\sigma_n\right) - \Phi(x)\right| \longrightarrow 0$

Convergence rate of SLLN

Theorem 8.10.

For any $q \geq 2$ and $q/2 < p \leq q$, if

$$\lim_{A \to \infty} \lim_{\epsilon \to 0^+} \epsilon^{r(p-1)} \sum_{n > A/\epsilon^r} n^{p-2} \sum_{i=1}^{n} \mathrm{P}\left(|X_i| \geq \epsilon \sigma_n n^{1/r}\right) = 0$$

where $r = 2q/(2p-q)$, then we have

$$\lim_{\epsilon \to 0^+} \epsilon^{r(p-1)} \sum_{n=1}^{\infty} n^{p-2} \mathrm{P}\left(|S_n| \geq \epsilon \sigma_n n^{1/r}\right) = \frac{2^{r(p-1)/2} \Gamma[\{1 + r(p-1)\}/2]}{(p-1)\Gamma(1/2)}$$

- Describes the decay of the probability $P(|\sigma_n^{-1} S_n| \geq \epsilon n^{1/r})$
- In the case that $p = q = r = 2$,and $E(X_i^2)$ is a constant, we have $\lim_{\epsilon \to 0^+} \epsilon^2 \sum_{n=1}^{\infty} \mathrm{P}\left(|S_n| \geq C \epsilon n\right) = 1$.

Convergence Rate with LIL

Let $W_n, n \geq 1$, be a nondecreasing sequence of positive random variables (i.e., $0 < W_1 \leq W_2 \leq \cdots$) and $Z_n, n \geq 1$, be a sequence of nonnegative random variables. Suppose that both sequences are predictable with respect to $\mathcal{F}_n, n \geq 1$. Define $\phi(t) = \sqrt{2t \log \log t}$ if $t > e$ and $\phi(t) = 1$ otherwise. Suppose that $W_n \xrightarrow{\text{a.s.}} \infty$ and $W_n / W_{n+1} \xrightarrow{\text{a.s.}} 1$ as $n \to \infty$ and that the following conditions are satisfied:

$$\frac{1}{\phi\left(W_n^2\right)} \sum_{i=1}^{n} \left[X_i 1_{(|X_i| > Z_i)} - \mathrm{E}\left\{X_i 1_{(|X_i| > Z_i)} \mid \mathcal{F}_{i-1}\right\}\right] \xrightarrow{\text{a.s.}} 0$$

$$\frac{1}{W_n^2} \sum_{i=1}^{n} \operatorname{var}\left\{X_i 1_{(|X_i| \leq Z_i)} \mid \mathcal{F}_{i-1}\right\} \xrightarrow{\text{a.s.}} 1$$

$$\sum_{i=1}^{\infty} \frac{1}{W_i^4} \mathrm{E}\left\{X_i^4 1_{(|X_i| \leq Z_i)} \mid \mathcal{F}_{i-1}\right\} < \infty \text{ a.s.}$$

Then we have $\limsup S_n / \phi\left(W_n^2\right) = 1$ a.s. and $\liminf S_n / \phi\left(W_n^2\right) = -1$ a.s.

8.7 Invariance Principles for Martingales

Invariance Principle

Let

$$\xi_n(t) = \frac{1}{U_n}\left(S_i + \frac{tU_n^2 - U_i^2}{X_{i+1}}\right) \text{ for } \frac{U_i^2}{U_n^2} \le t < \frac{U_{i+1}^2}{U_n^2}$$

$0 \le i \le n-1$, and $\xi_n(1) = U_n^{-1} S_n$, where $U_0^2 = 0$ and $U_i^2 = \sum_{j=1}^{i} X_j^2$, $i \ge 1$
Intuitively, ξ_n is a function on $[0,1]$ obtained by linear interpolating between the points $\left(U_i^2/U_n^2, S_i/U_n\right)$, $i = 0, \ldots, n$.
Suppose that the following Lindeberg condition holds:

$$\frac{1}{s_n^2}\sum_{i=1}^{n} \mathrm{E}\left\{X_i^2 1_{(|X_i| > \epsilon s_n)}\right\} \longrightarrow 0$$

as $n \to \infty$ for every $\epsilon > 0$, where $s_n^2 = \mathrm{E}\left(S_n^2\right)$, and that

$$\frac{U_n^2}{s_n^2} \xrightarrow{\mathrm{P}} \eta^2$$

where the random variable η^2 is a.s. positive. Then $\xi_n \xrightarrow{\mathrm{d}} W$, where W is the Brownian motion on $[0,1]$.

Application

Example 8.19. Because $\xi_n \xrightarrow{\mathrm{d}} W$ implies that $h(\xi_n) \xrightarrow{\mathrm{d}} h(W)$ for any continuous function h on $\mathcal{C}$.

- Suppose $h(x) = x(1)$ for $x \in \mathcal{C}$, then we have a CLT for a martingale S_n normalized by U_n:

$$U_n^{-1} S_n \xrightarrow{\mathrm{d}} W(1) \sim N(0,1).$$

- If we let $h(x) = \sup_{t\in[0,1]} x(t)$ and notes that $\sup_{t\in[0,1]} \xi_n(t) = U_n^{-1} \max_{0\le i\le n} S_i$, then we have

$$U_n^{-1} \max_{0\le i\le n} S_i \xrightarrow{\mathrm{d}} \sup_{t\in[0,1]} W(t)$$

Extension

For a general sequence of random variables $0 < W_1 \leq W_2 \leq \dots$, let

$$\zeta_n(t) = \frac{1}{\phi\left(W_n^2\right)} \left(S_i + \frac{tW_n^2 - W_i^2}{W_{i+1}^2 - W_i^2} X_{i+1} \right)$$
$$\text{for } \frac{W_i^2}{W_n^2} \leq t < \frac{W_{i+1}^2}{W_n^2}$$

$0 \leq i \leq n-1$, and $\zeta_n(1) = \phi^{-1}\left(W_n^2\right) S_n$.
Let $\rho(x,y) = \sup_{t \in [0,1]} |x(t) - y(t)|$. And the space K of absolutely continuous functions x on $[0,1]$ with $x(0) = 0$.
Invariance principle in the LIL: ζ_n r.c. K w.r.t. ρ on $\mathcal{C}$ a.s. In words, we have with probability 1 that the sequence ζ_n is relative compact in $\mathcal{C}$ and its set of ρ limit points coincides with K.

8.8 Martingale Approximation

Definitions

Idea: to approximate a random process by a martingale, so that the desired limit theorem for this random process follows from that of the martingale.

The random process

- Let $\xi_i, r \in Z$ be a stationary and ergodic Markov chain where Z is the set of all integers.
- Let $X_i = g(\xi_i)$, where g is a measurable function.
- The random process of interest: $S_n = \sum_{i=1}^n X_i, n \geq 1$.

The approximation

- Let $\mathcal{F}_k = \sigma(\xi_j, j \leq k)$.
- The projection: $\mathcal{P}_k Z = \mathrm{E}(Z \mid \mathcal{F}_k) - \mathrm{E}(Z \mid \mathcal{F}_{k-1})$.
- $D_k = \sum_{i=k}^{\infty} \mathcal{P}_k g(\xi_i)$. Then $D_k, \mathcal{F}_k, k \in Z$, is a sequence of martingale differences
- The approximating martingale: $M_n = \sum_{k=1}^n D_k, \mathcal{F}_n, n \geq 1$.

Bound for $S_n - M_n$

Let $\delta_{i,q} = \|\mathcal{P}_0 g(\xi_i)\|_q$, where for any random variables Z and $q > 0$, $\|Z\|_q = \{\mathrm{E}(|Z|^q)\}^{1/q}$, and $\Delta_{j,q} = \sum_{i=j}^{\infty} \delta_{i,q}$

Corollary 8.4.

Let $\mathrm{E}\{g(\xi_0)\} = 0$ and $g(\xi_0) \in L^q$ for some $q > 1$. We have $S_n - M_n = o\left(n^{1/q}\right)$ a.s., provided that $\Delta_{0,q} < \infty$ and

$$\sum_{j=1}^{\infty} j^{-a} \Delta_{j,q}^{b} < \infty$$

where $a = \{(q+4)/2(q+1)\} \wedge 1$ and $b = q/(q+1)$. Here, $S_n - M_n = o\left(n^{1/q}\right)$ a.s. means that $(S_n - M_n)/n^{1/q} \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$.

SLLN of S_n

A function h is slowly varying if for any $\lambda > 0, \lim_{x \to \infty} h(\lambda x)/h(x) = 1$.

SLLN

Let h be a positive, nondecreasing slowly varying function.
(i) If $q > 2, \Delta_{n,q} = O\left[(\log n)^{-\alpha}\right]$ for some $0 \le \alpha \le 1/q$, and

$$\sum_{j=1}^{\infty} \left\{ j^{\alpha} h\left(2^j\right) \right\}^{-q} < \infty$$

then $S_n/\sqrt{n}h(n) \xrightarrow{\text{a.s.}} 0$, as $n \to \infty$.

SLLN of S_n

SLLN

(ii) If $1 < q \leq 2, \Delta_{0,q} < \infty$, and

$$\sum_{j=1}^{\infty} \left\{ h\left(2^j\right) \right\}^{-q} < \infty$$

then $S_n / n^{1/q} h(n) \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$.
(iii) If $1 < q < 2$ and the assumption of Corollary 8.4 holds, then $S_n / n^{1/q} \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$.

The results of LIL and strong invariance principle can be obtained similarly.

References

Jiming Jiang (2010)
Large Sample Techniques for Statistics
vol. 0 (Springer New York, 2010).

Thank You!