Probability with Martingales
Chapter 5: Integration

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Outline

5.1 Notations

Let (S, Σ, μ) be a measure space and let $f \in m\Sigma$. The (Lebesgue) integral of f with respect to μ is denoted as

$$\mu(f) :=: \int_S f(s)\mu(ds) :=: \int_S f \; d\mu$$

The integral on set $A \in \Sigma$ is denoted as

$$\int_A f(s)\mu(ds) :=: \int_A f \; d\mu :=: \mu(f; A) := \mu(f I_A)$$

eg.

$$\mu(f; f \geq x) := \mu(f; A), \text{ where } A = \{s \in S : f(s) \geq x\}$$

5.2.1 Definition

Recall $f \in (m\Sigma)^+$ is called simple if it can be written as a finite sum

$$f = \sum_{k=1}^{m} a_k I_{A_k}$$

where $a_k \in [0, \infty]$ and $A_k \in \Sigma$. We write $f \in SF^+$.

Definition: Integral of non-negative simple functions

If $A \in \Sigma$, we define

$$\mu_0(I_A) := \mu(A) \leq \infty$$

The integral of $f \in SF^+$ is then defined as

$$\mu_0(f) = \sum a_k \mu(A_k) \leq \infty$$

5.2.2 Properties

1. $\mu_0(f)$ is well-defined
2. If $f, g \in SF^+$ and $f = g$ a.e. then $\mu_0(f) = \mu_0(g)$
3. ('Linearity') If $f, g \in SF^+$ and $c \geq 0$ then $f + g$ and cf are in SF^+, and

$$\mu_0(f + g) = \mu_0(f) + \mu_0(g), \mu_0(cf) = c\mu_0(f)$$

4. (Monotonicity) If $f, g \in SF^+$ and $f \leq g$ then $\mu_0(f) \leq \mu_0(g)$
5. If $f, g \in SF^+$ then $f \bigwedge g$ and $f \bigvee g$ are in SF^+

5.3.1 Definition

Definition: Integral of non-negative Σ-measurable functions

For $f \in (m\Sigma)^+$, we define

$$\mu(f) := sup\{\mu_0(h) : h \in SF^+, h \leq f\} \leq \infty$$

For $A \in \Sigma$, we further define

$$\int_A f \, d\mu =: \mu(f; A) := \mu(fI_A)$$

Clearly, $\mu(f) = \mu_0(f), \forall f \in SF^+$.

5.3.2 Monotone-Convergence Theorem(MON)

Lemma

If $f \in (m\Sigma)^+$ and $\mu(f) = 0$ then

$$\mu(\{f > 0\}) = 0$$

Proof:
$\{f > 0\} =\uparrow \lim\{f > \frac{1}{n}\}$.
Then by the monotone convergence property of measures,
$\mu(\{f > 0\}) > 0 \implies \uparrow \lim \mu\{f > \frac{1}{n}\} > 0$
$\implies \exists n$ s.t. $\mu(\{f > \frac{1}{n}\}) > 0$
Then $\mu(f) \geq \mu_0(\frac{1}{n} I_{\{f > \frac{1}{n}\}}) = \frac{1}{n}\mu(\{f > \frac{1}{n}\}) > 0$.
This contradicts with $\mu(f) = 0$. $\square$

5.3.2 Monotone-Convergence Theorem(MON)

Theorem: Monotone Convergence

If (f_n) is a sequence of elements of $(m\Sigma)^+$ such that $f_n \uparrow f$, then

$$\mu(f_n) \uparrow \mu(f) \leq \infty$$

i.e.

$$\int_S \uparrow \lim f_n(s) d\mu = \uparrow \lim \int_S f_n(s) d\mu$$

5.3.2 Monotone-Convergence Theorem(MON)

Proof

$f \geq f_n \implies \int_S f(s)d\mu \geq \int_S f_n(s)d\mu$. Taking limit on n we get

$$\int_S f(s)d\mu \geq\uparrow \lim \int_S f_n(s)d\mu$$

It remains to prove that $\int_S f \leq\uparrow \lim \int_S f_n$.
Let $h \in SF^+$ such that $h \leq f :=\uparrow \lim f_n = \sup_n f_n$. Then

$$h(s) \leq \sup_n f_n(s), \forall s \in S$$

Therefore, let $\epsilon > 0$, we have

$$\forall s \in S, \exists N = N(s) \text{ s.t. } f_N(s) \geq (1-\epsilon)h(s)$$

Since (f_n) is increasing, we have $f_n(s) \geq (1-\epsilon)h(s), \forall n \geq N$.

Proof (Continued)

For each n, define

$$E_n := \{s \in S : f_n(s) \geq (1-\epsilon)h(s)\}$$

Then $f_n \geq (1-\epsilon)h$ on E_n.
And $E_1 \subset E_2 \subset \dots$ and $\bigcup_{n=1}^{\infty} E_n = S$. We have

$$(1-\epsilon)\int_{E_n} h = \int_{E_n}(1-\epsilon)h \leq \int_{E_n} f_n = \int_S f_n I_{E_n}$$

Since $f_n I_{E_n} \leq f_n$, we have $\int_{E_n} f_n \leq \int_S f_n$. As ϵ is arbitrary, we have

$$\int_{E_n} h \leq \int_S f_n$$

Proof (Continued)

Taking limit on n, we have

$$\sup_n \int_{E_n} h \leq\uparrow \lim \int_S f_n$$

Recall $h \in SF^+$ and $h \leq f$. Hence we can write $h = \sum_{k=1}^{m} a_k I_{A_k}$ where $a_k \in [0, \infty]$ and $A_k \in \Sigma$ and $\int_S h = \sum_{k=1}^{m} a_k \mu(A_k)$.

We have

$$\int_{E_n} h = \int_{E_n} \sum_{k=1}^{m} a_k I_{A_k} = \int_S \sum_{k=1}^{m} a_k I_{A_k \cap E_n} = \sum_{k=1}^{m} a_k \mu(A_k \cap E_n)$$

Since $\sup_n \mu(A_k \cap E_n) = \mu(A_k)$, we have $\int_S f = \sup_n \int_{E_n} h$. $\square$

5.3.2 Monotone-Convergence Theorem(MON)

Application: To explicitly find a sequence of simple functions that converges to f.

Define the r^{th} **staircase function** $\alpha^{(r)} : [0, \infty] \to [0, \infty]$ as follows:

$$\alpha^{(r)}(x) := \begin{cases} 0 & \text{if } x = 0, \\ (i-1)2^{-r} & \text{if } (i-1)2^{-r} < x \le i2^{-r} \le r \ (i \in \mathbb{N}), \\ r & \text{if } x > r. \end{cases}$$

Let $f^{(r)} := \alpha^{(r)} \circ f$. Then $f^{(r)} \in SF^{+}$, and $f^{(r)} \uparrow f (\alpha^{(r)} \uparrow id_{[0,\infty]})$.
By (MON), $\mu(f) = \uparrow \lim \mu(f^{(r)}) = \uparrow \lim \mu_0(f^{(r)})$.

5.3.3 Properties

Suppose $f, g \in (m\Sigma)^+$ and $\alpha, \beta \in \mathbb{R}^+$, then

1. If $f = g$ a.e., then $\mu(f) = \mu(g)$
   Proof: Let $f^{(r)} = \alpha^{(r)} \circ f, g^{(r)} = \alpha^{(r)} \circ g$.
   Then $f^{(r)} = g^{(r)}$ a.e., and so $\mu(g^{(r)}) = \mu(f^{(r)})$.
   The result follows from (MON).
2. If (f_n) is a sequence in $(m\Sigma)^+$ that converges to f except on a μ-null set N ($f_n \uparrow f$ a.e.). Then $\mu(f_n) \uparrow \mu(f)$.
   Proof: $\mu(f) = \mu(fI_{S\setminus N})$ and $\mu(f_n) = \mu(f_n I_{S\setminus N})$.
   $f_n I_{S\setminus N} \uparrow f I_{S\setminus N}$ everywhere. The result follows from (MON).
3. ('Linearity') $\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$
   Proof: Take sequences (f_n) and (g_n) of elements from SF^+, such that $f_n \uparrow f$ and $g_n \uparrow g$. Then $(\alpha f_n + \beta g_n) \uparrow (\alpha f + \beta g)$.
   We know $\mu(\alpha f_n + \beta g_n) = \alpha\mu(f_n) + \beta\mu(g_n)$.
   By (MON), LHS $= \uparrow \lim \mu(\alpha f_n + \beta g_n)$ and
   RHS $= \uparrow \lim \alpha\mu(f_n) + \uparrow \lim \beta\mu(g_n)$.

5.3.4 The Fatou Lemmas

Lemma 1: FATOU

For a sequence (f_n) in $(m\Sigma)^+$, we have

$$\mu(\liminf f_n) \leq \liminf \mu(f_n)$$

Proof

Let $f = \liminf f_n$ and $g_k := \inf_{n \geq k} f_n$. By definition, $f = \uparrow \lim g_k$.
Applying (MON) we get $\mu(f) = \uparrow \lim_k \mu(g_k)$.
For $n \geq k$, we have $f_n \geq g_k$, so that $\mu(f_n) \geq \mu(g_k)$, then

$$\mu(g_k) \leq \inf_{\{n : n \geq k\}} \mu(f_n)$$

Hence $\mu(f) = \uparrow \lim_k \mu(g_k) \leq \uparrow \lim_k \inf_{\{n : n \geq k\}} \mu(f_n) =: \liminf \mu(f_n)$

5.3.4 The Fatou Lemmas

Lemma 2: Reverse Fatou Lemma

For a sequence (f_n) in $(m\Sigma)^+$ such that $f_n \leq g, \forall n$ for some $g \in (m\Sigma)^+$ and $\mu(g) < \infty$, we have

$$\mu(\limsup f_n) \geq \limsup \mu(f_n)$$

Proof

Apply (FATOU) to the sequence $(g - f_n)$, we get

$$\mu(\liminf (g - f_n)) \leq \liminf \mu(g - f_n)$$

Then $\mu(g - \limsup(f_n)) \leq \liminf(\mu(g) - \mu(f_n))$.
Thus $\mu(g) - \mu(\limsup(f_n)) \leq -\limsup(\mu(f_n) - \mu(g))$.
The result immediately follows.

5.4.1 Definitions

Definition: Positive and Negative Parts of f

Suppose $f \in m\Sigma$. Let $f^+(s) := \max(f(s), 0)$ and $f^-(s) := \min(f(s), 0)$.
Then $f^+, f^- \in (m\Sigma)^+$ and $f = f^+ - f^-, |f| = f^+ + f^- \in (m\Sigma)^+$

Definition: Integrable Function

For $f \in m\Sigma$, we say that f is μ-integrable and write

$$f \in \mathcal{L}^1(S, \Sigma, \mu)$$

if

$$\mu(|f|) = \mu(f^+) + \mu(f^-) < \infty$$

Let $\mathcal{L}^1(S, \Sigma, \mu)^+$ denote the non-negative elements in $\mathcal{L}^1(S, \Sigma, \mu)$.

5.4.1 Definitions

Definition: Integral of Integrable Functions

For $f \in \mathcal{L}^1(S, \Sigma, \mu)$, define

$$\int f \, d\mu := \mu(f) := \mu(f^+) - \mu(f^-)$$

5.4.2 Properties

Let $f, g \in \mathcal{L}^1(S, \Sigma, \mu)$ and $\alpha, \beta \in \mathbb{R}$, then

1. ('Triangle Inequality') $|\mu(f)| \leq \mu(|f|)$
   Proof: LHS$=|\mu(f^+) - \mu(f^-)|$ and RHS$= \mu(f^+) + \mu(f^-)$
2. (Linearity) $\alpha f + \beta g \in \mathcal{L}^1(S, \Sigma, \mu)$ and

$$\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$$

   Proof: Similar to the proof in section 5.3.3

5.4.3 Dominated-Convergence Theorem(DOM)

Theorem: Dominated Convergence

Suppose $f_n, f \in m\Sigma$ such that $f_n(s) \to f(s), \forall s \in S$, and that the sequence (f_n) is **dominated** by an element g of $\mathcal{L}^1(S, \Sigma, \mu)^+$ with $\mu(g) < \infty$:

$$|f_n(s)| \leq g(s), \forall s \in S, \forall n \in N$$

Then

$$f_n \to f \text{ in } \mathcal{L}^1(S, \Sigma, \mu) : \mu(|f_n - f|) \to 0$$

hence $\mu(f_n) \to \mu(f)$

5.4.3 Dominated-Convergence Theorem(DOM)

Proof

$\lim |f_n - f| = 0$
Since $f_n \to f$ pointwisely, we know $|f(s)| \leq g(s), \forall s \in S$. Then $|f_n - f| \leq 2g < \infty$.
Then by Reverse Fatou Lemma,

$$\limsup \mu(|f_n - f|) \leq \mu(\limsup |f_n - f|) = \mu(0) = 0$$

Hence $\limsup \mu(|f_n - f|) = 0 \leq \liminf \mu(|f_n - f|)$.
Therefore,

$$\lim \mu(|f_n - f|) = 0$$

and

$$0 \leq |\mu(f_n) - \mu(f)| = |\mu(f_n - f)| \leq \mu(|f_n - f|)$$

By squeeze theorem, $\mu(f_n) \to \mu(f)$.

5.4.4 Scheffé's Lemma(SCHEFFÉ)

Scheffé's Lemma: Part 1

Suppose $f_n, f \in \mathcal{L}^1(S, \Sigma, \mu)^+$ such that $f_n \to f$ a.e., then

$$\mu(|f_n - f|) \to 0 \text{ if and only if } \mu(f_n) \to \mu(f)$$

Scheffé's Lemma: Part 2

Suppose $f_n, f \in \mathcal{L}^1(S, \Sigma, \mu)$ such that $f_n \to f$ a.e., then

$$\mu(|f_n - f|) \to 0 \text{ if and only if } \mu(|f_n|) \to \mu(|f|)$$

Proof of Part 1

($\Rightarrow$): Trivial
($\Leftarrow$): Suppose now that $\mu(f_n) \to \mu(f)$
Recall $\mu(|f_n - f|) = \mu((f_n - f)^+) + \mu((f_n - f)^-)$.
Since $(f_n - f)^- \leq f$ ($(f_n - f)^- > 0$ when $f_n < f$) and $(f_n - f)^- \to 0$ a.e., by (DOM),

$$\mu((f_n - f)^-) \to 0$$

$\mu((f_n - f)^+) = \mu(f_n - f; f_n \geq f) = \mu(f_n) - \mu(f) - \mu(f_n - f; f_n < f)$
And

$$|\mu(f_n - f; f_n < f)| \leq |\mu((f_n - f)^-)| \to 0$$

□

5.5.1 The Standard Machine

Procedure to prove that a result is true for all functions h in a space such as $\mathcal{L}^1(S, \Sigma, \mu)$:

- Step 1. Show the result is true when h is an indicator function.
- Step 2.Using linearity, prove the result for h in SF^+.
- Step 3. Use (MON) to obtain the result for $h \in (m\Sigma)^+$ (integrability is usually unnecessary)
- Step 4. Prove the result by writing $h = h^+ - h^-$ and using linearity

Be careful with the commutative property of operations: what operations are commutative under what conditions
eg. $\int, \lim, \sup/\inf, \cap/\cup, \sum$

5.5.2 Integrals Over Subsets of σ-algebra

For $f \in \mathcal{L}^1(S, \Sigma, \mu)$ and $A \in \Sigma$, let $f|_A$ denote the restriction of f on A, Σ_A denote the σ-algebra of subsets of A that belong to Σ, and μ_A denote the measure which is μ restricted to (A, Σ_A).
Then we have

$$\mu_A(f|_A) = \mu(f; A)$$

where $\mu(f; A) := \mu(fI_A)$.
The proof can be done with the standard machine.

5.5.3 The Measure $f\mu$

Let $f \in (m\Sigma)^+, A \in \Sigma$. We define (should distinguish with $\mu(A)$):

$$(f\mu)(A) := \mu(f; A) := \mu(fI_A)$$

Notice that $(f\mu)$ is a measure on (S, Σ). For $h \in (m\Sigma)^+$, define $(h(f\mu))(A) := (f\mu)(hI_A)$. Then we can use the standard machine to prove

$$(h(f\mu))(A) = ((fh)\mu)(A)$$

Furthermore, if $h \in (m\Sigma)$, then

$$h \in \mathcal{L}^1(S, \Sigma, f\mu) \text{ iff } fh \in \mathcal{L}^1(S, \Sigma, \mu)$$

and in this case $(f\mu)(h) = \mu(fh)$

5.5.3 The Measure $f\mu$

Definition

With previous assumptions, we say that λ has density f relative to μ if $\lambda = f\mu$. And we write

$$d\lambda/d\mu = f$$

For $F \in \Sigma$, if λ has density relative to μ, then

$$\mu(F) = 0 \implies \lambda(F) = 0$$

The converse is true under certain conditions.

Radon-Nikodým Theorem (proved in Chapter 14)

If μ and λ are σ-finite measures on (S, Σ) such that $\mu(F) = 0 \implies \lambda(F) = 0$, then $\lambda = f\mu$ for some $f \in (m\Sigma)^+$.